Math 105 - Assignment 1 : Solutions

1. a) Recall the equation of plane that through a point $\vec{x}_0 = (x_0, y_0, z_0)$ and normal $\vec{n} = (a,b,c)$ is given by:

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_0 \quad , \quad \vec{x} = (x,y,z). \quad (\star)$$

So we need to find a $\vec{n}$ perpindicular to Q and $\vec{x}_0$ on Q, and then plug it into $(\star)$.

Now we know that Q, P are parallel, thus the normal of P is parallel to the normal of Q. Recall if you have a plane given by the equation:

$$ax + by + cz = d,$$

then the normal vector is (a,b,c). So in our case P is given by the equation:

$$2x - y + 6z = 4$$

So $\vec{n} = (2,-1,6)$ is perpindicular to P and hence Q. Also Q passes through $(1,2,-4)$ so $\vec{x}_0 = (1,2,-4)$, and we plug into $(\star)$.

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_0$$

$$\Rightarrow (2,-1,6) \cdot (x,y,z) = (2,-1,6) \cdot (1,2,-4)$$

$$\Rightarrow 2x - y + 6z = (2)(1) + (-1)(2) + (6)(-4)$$

$$\Rightarrow \boxed{2x - y + 6z = -24}$$

b) In part a) we showed that $\vec{n} = (2,-1,6)$ was perpindicular to P. All the vectors that are perpindicular to P are going to be of the form $c\vec{n}$ (ie. stretching/compression of $\vec{n}$).

So we want to find c such that

$$\|c\vec{n}\| = 5.$$

Now note:

$$5 = \|c\vec{n}\| = |c|\,\|\vec{n}\| \quad , \quad \text{since } \|c\vec{x}\| = |c|\,\|\vec{x}\|$$

$$\Rightarrow \quad |c| = \frac{5}{\|\vec{n}\|}$$

$$= \frac{5}{\sqrt{\vec{n}\cdot\vec{n}}}$$

$$= \frac{5}{\sqrt{2^2 + (-1)^2 + 6^2}}$$

$$= \frac{5}{\sqrt{41}}$$

$$\Rightarrow \quad c = \pm \frac{5}{\sqrt{41}}$$

So the only vectors of legnth 5 that are perpindicular to P are,

$$\boxed{\begin{aligned} &\frac{5}{\sqrt{41}}(2,-1,6) = \left(\frac{10}{\sqrt{41}}, \frac{-5}{\sqrt{41}}, \frac{30}{\sqrt{41}}\right) \\ -&\frac{5}{\sqrt{41}}(2,-1,6) = \left(\frac{-10}{\sqrt{41}}, \frac{5}{\sqrt{41}}, \frac{-30}{\sqrt{41}}\right) \end{aligned}}$$

2. $$f(x,y) = \frac{\log(y+1)}{(x^2-y^2)(1-\sin x)}$$

- First note that $\log(y+1)$ is only defined when $y+1>0$. So we already know that $y>-1$.

We also cannot have the denominator equal zero. The denominator is zero when

① $x^2-y^2=0$

② $1-\sin x=0$

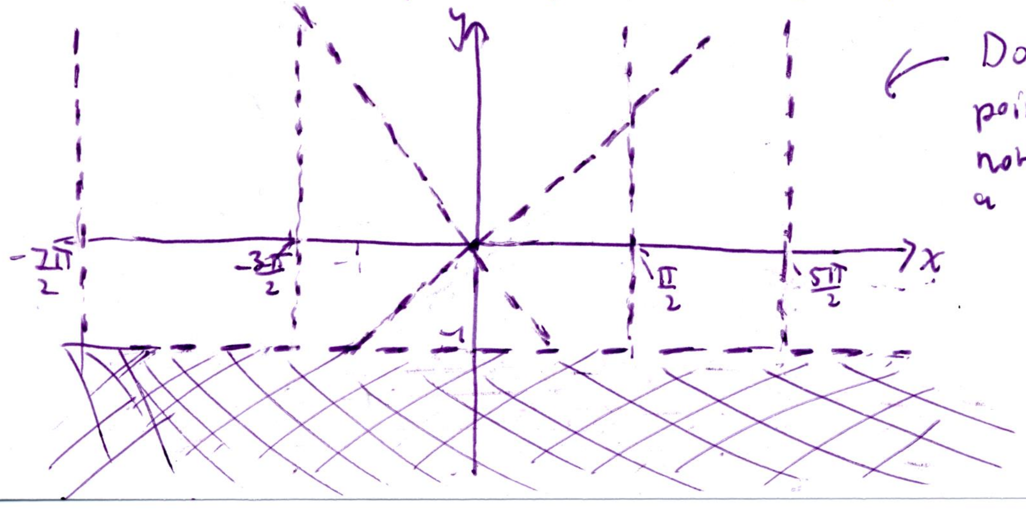
- In the case of ① we hav $x^2=y^2$, by taking square roots we get:

$$|x|=|y| \Rightarrow y=\pm|x| \qquad (\text{Note } \sqrt{x^2}=|x|, \underline{\text{not}}\ x)$$

- In the case of 2, we have $\sin x = 1$. This is true when

$$x=\frac{\pi}{2}+2k\pi, \quad k \text{ is an integer.}$$

So $D(f)=\{(x,y)\in\mathbb{R}^2 \mid y>-1,\ y\neq\pm x,\ x\neq\frac{\pi}{2}+2k\pi\}$



Domain is all the points that are not shaded or have a line.

3. $z = f(x,y) = (x^2-2x+4y^2-3)\log(x^2+y^2)$

The level set when $z=0$ is given by all the points $(x,y) \in \mathbb{R}^2$ such that:

$$0 = (x^2-2x+4y^2-3)\log(x^2+y^2) \quad (*)$$

$(*)$ is true if either:

① $x^2-2x+4y^2-3=0$

② $\log(x^2+y^2)=0$

Lets deal with each case separately.

① Before we begin, lets complete the square:

$$x^2-2x = x^2-2x+1-1$$
$$= (x-1)^2-1$$

So we have

$$x^2-2x+4y^2-3=0$$
$$\Rightarrow (x-1)^2-1+4y^2-3=0$$
$$\Rightarrow (x-1)^2+4y^2=4$$

This is one equation of an ellipse centered at $(1,0)$. Note the equation is symetric about y, it is enough to graph the upper half and flip about the line $y=0$ to get the bottom half. The top half of the ellipse is given by

$$y = \sqrt{1-\tfrac{1}{4}(x-1)^2}$$

y is a maximum when $\frac{1}{4}(x-1)^2=0 \Rightarrow x=1$

And y gets small as x goes to the left or right of 1.

Also the part inside the square root cannot be negative and thus y is a minimum when $1-\frac{1}{4}(x-1)^2=0$

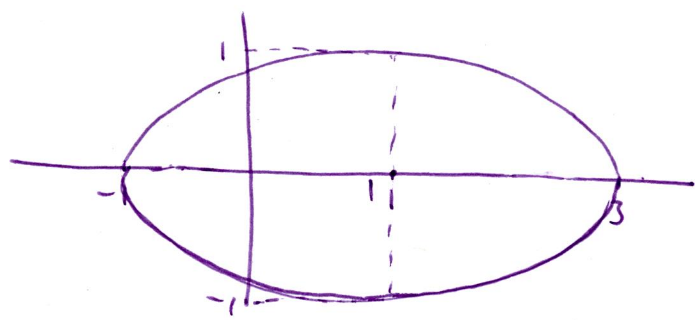
$$1-\frac{1}{4}(x-1)^2=0$$

$$\Rightarrow \quad (x-1)^2=4$$

$$\Rightarrow \quad |x-1|=2$$

$$\Rightarrow \quad x=3,\ -1$$

So $(x-1)^2+4y^2=4$ is

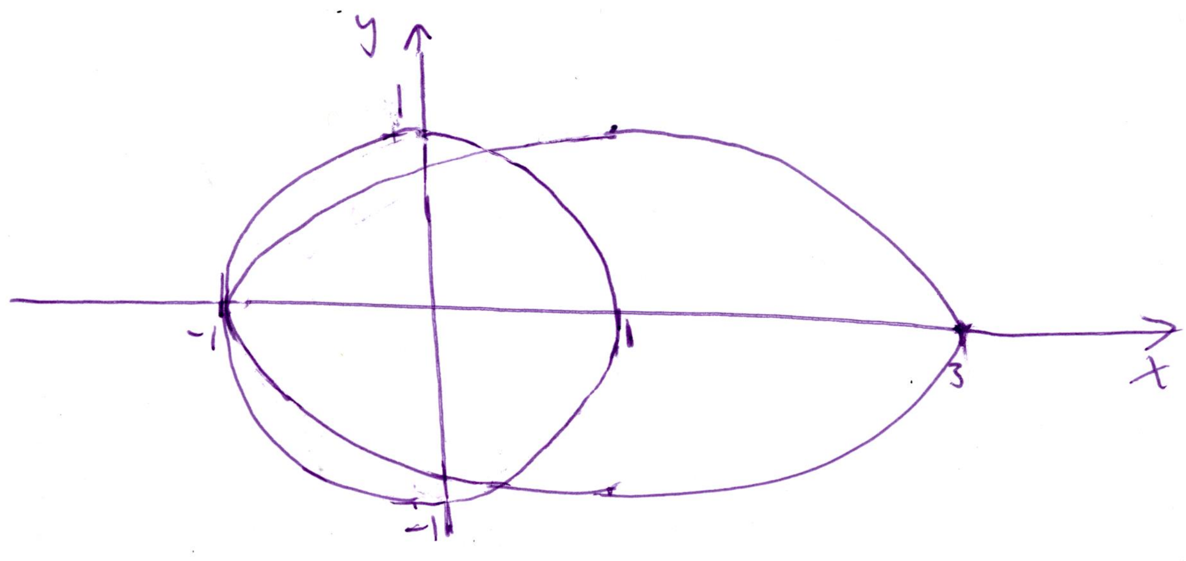


② $\log(x^2+y^2)=0$

$\Rightarrow \quad x^2+y^2=1$

Which is the circle of radius 1 centered at $(0,0)$.

Putting ①, ② together we get



4. $4x^2 - 9(y-1)^2 - z = 0$

Let use do each of these traces separately:

① xy-trace. This is the trace of $z=0$

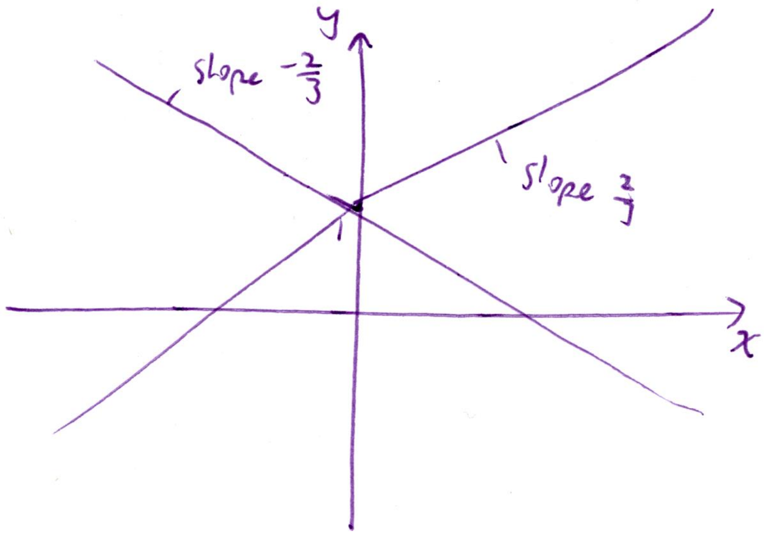
$\Rightarrow \quad 4x^2 - 9(y-1)^2 = 0$

$\Rightarrow \quad 4x^2 = 9(y-1)^2$

$\Rightarrow \quad 2|x| = 3|y-1|$

$\Rightarrow \quad y-1 = \pm \frac{2|x|}{3}$

$\Rightarrow \quad y = 1 \pm \frac{2}{3}|x|$

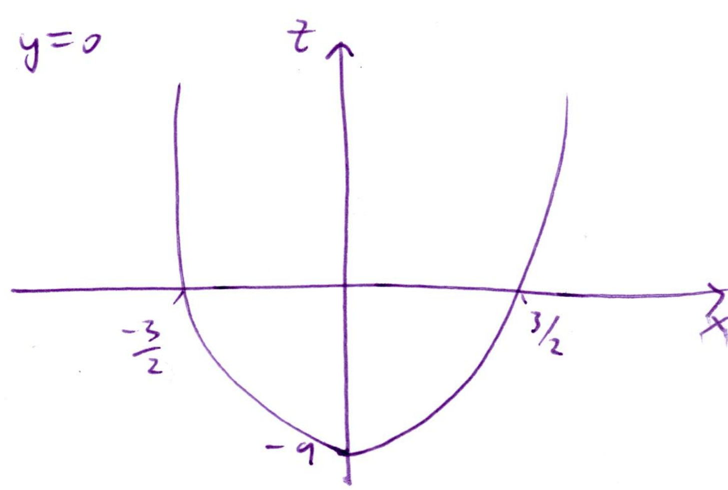


② xz-trace. This is the trace of $y=0$

$\Rightarrow \quad 4x^2 - 9(0-1)^2 - z = 0$

$\Rightarrow \quad z = 4x^2 - 9$

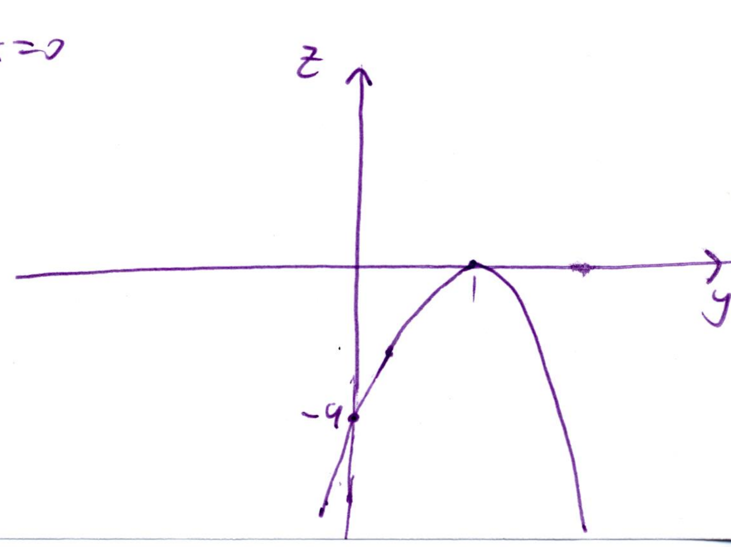
$= (2x-3)(2x+3)$



③ yz-trace, This is the trace of $x=0$

$\Rightarrow \quad 4(0)^2 - 9(y-1)^2 - z = 0$

$\Rightarrow \quad z = -9(y-1)^2$



Notice that $z = 4x^2 - 9(y-1)^2$

If we look at the trace $x=c$, we get

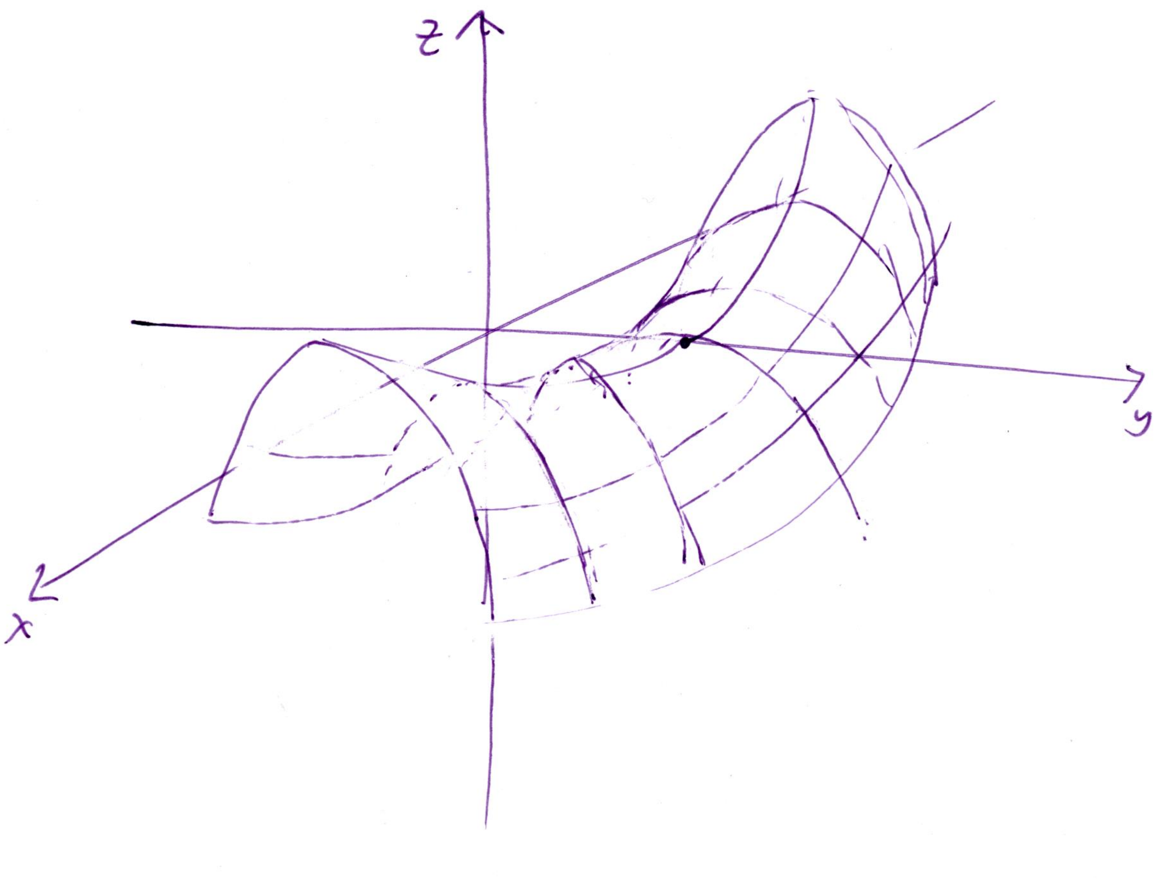
$$z = 4c^2 - 9(y-1)^2$$

which is a parabola facing downwards with a maximum at $y=1$. Also as $|c|$ gets larger the parabola increases in height.

Similarly if we look at the trace $y=c$, we get

$$z = 4x^2 - 9(c-1)^2$$

This is a parabola facing upwards with a minimum at $x=0$. Also as c deviates from 1 the parabola decreases in height.



5. a) $P(t) = \|t\vec{x} + \vec{y}\|^2$

$\Rightarrow$ $P(t) = (t\vec{x}+\vec{y}) \cdot (t\vec{x}+\vec{y})$, since $\|\vec{x}\|^2 = \vec{x}\cdot\vec{x}$

$= (t\vec{x})\cdot(t\vec{x}) + (t\vec{x})\cdot\vec{y} + \vec{y}\cdot(t\vec{x}) + \vec{y}\cdot\vec{y}$

$= t^2\,\vec{x}\cdot\vec{x} + t\,\vec{x}\cdot\vec{y} + t\,\vec{y}\cdot\vec{x} + \vec{y}\cdot\vec{y}$ by linearity of dot product.

$= t^2\,\vec{x}\cdot\vec{x} + 2t\,\vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{y}$, since $\vec{x}\cdot\vec{y} = \vec{y}\cdot\vec{x}$

$= \|\vec{x}\|^2 t^2 + 2\,\vec{x}\cdot\vec{y}\,t + \|\vec{y}\|^2$, since $\|\vec{x}\|^2 = \vec{x}\cdot\vec{x}$.

b) $P(t)$ is quadradic by part a). So

$P'(t) = 2\|\vec{x}\|^2 t + 2\,\vec{x}\cdot\vec{y} = 0$

$\Rightarrow$ $t = \dfrac{-\vec{x}\cdot\vec{y}}{\|x\|^2}$

$P''(t) = 2\|x\|^2 > 0$ since $\|x\|^2 > 0$

So we have $P(t)$ has a minimum at $t^* = -\dfrac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|^2}$

So the minimum of $P(t)$ is $P(t^*)$.

$P(t^*) = P\left(-\dfrac{\vec{x}\cdot\vec{y}}{\|x\|^2}\right)$

$= \|\vec{x}\|^2\left(-\dfrac{\vec{x}\cdot\vec{y}}{\|x\|^2}\right)^2 + 2\,\vec{x}\cdot\vec{y}\left(-\dfrac{\vec{x}\cdot\vec{y}}{\|x\|^2}\right) + \|\vec{y}\|^2$

$= \dfrac{|\vec{x}\cdot\vec{y}|^2}{\|\vec{x}\|^2} - 2\dfrac{|\vec{x}\cdot\vec{y}|^2}{\|x\|^2} + \dfrac{\|\vec{x}\|^2\|\vec{y}\|^2}{\|x\|^2}$

$$= \frac{|\vec{x}\cdot\vec{y}|^2 - 2|\vec{x}\cdot\vec{y}|^2 + \|\vec{x}\|^2\|\vec{y}\|^2}{\|\vec{x}\|^2}$$

$$= \frac{\|\vec{x}\|^2\|\vec{y}\|^2 - |\vec{x}\cdot\vec{y}|}{\|\vec{x}\|^2},$$

As required.

c) Note that $\|\vec{x}+\vec{y}\|^2 = p(1)$

So $\|\vec{x}+\vec{y}\|^2 = p(1)$

$= |p(1)|$ since p is positive

$= \left|\, \|\vec{x}\|^2 + 2\,\vec{x}\cdot\vec{y} + \|\vec{y}\|^2 \right|$, by a)

$\leq \|\vec{x}\|^2 + 2|\vec{x}\cdot\vec{y}| + \|\vec{y}\|^2$, by triangle inequality

$\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2$, by Cauchy-Schwarz inequaility ie b)

$= (\|\vec{x}\| + \|\vec{y}\|)^2$

By taking square roots we get:

$\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|^2$